Do not use the Two sample-t-test any more!

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against a two-sided alternative hypothesis:

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t-Test

Given 2 independent random samples

$$\boldsymbol{y}_i^T = (\boldsymbol{y}_{i1}, \boldsymbol{y}_{i2}, \ldots, \boldsymbol{y}_{in_i}),\ n_i;\ i = 1, 2$$

distributed as $N(\vec{\mu}_i; \sigma^2 I_{n_i}); \quad \sigma^2 > 0$

Then

$$\boldsymbol{s}^2 = \frac{\boldsymbol{SQ}_{y_1} + \boldsymbol{SQ}_{y_2}}{n_1 + n_2 - 2}$$

is the pooled estimator of σ^2.

t-Test

From this it follows that

$$\boldsymbol{t} = \frac{\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_2}{\boldsymbol{s}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

is distributed as

$$t\left(n_1 + n_2 - 2; \lambda = \frac{\mu_1 - \mu_2}{\sigma} \sqrt{\frac{n_1 n_2}{n_1 + n_2}}\right).$$

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A uniformely best unbiased α-test for all $0 < \alpha < 1$ is

$$k\begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} = \begin{matrix} 1, \text{ for } |t| > t(n_1 + n_2 - 2; 1 - \frac{\alpha}{2}) \\ 0, \text{ otherwise} \end{matrix}$$

Welch-Test

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In the case of unequal variances Welch(1947) and Trickett and Welch(1954) proposed an approximate test based on

$$\boldsymbol{t}^* = \frac{\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_2}{\sqrt{\frac{\boldsymbol{s}_1^2}{n_1} + \frac{\boldsymbol{s}_2^2}{n_2}}}$$

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with:

$$f = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{s_1^4}{(n_1-1)n_1^2} + \frac{s_2^4}{(n_2-1)n_2^2}}$$

Which Test?

Hence, many theoretical statisticians nowadays do not recommend pre-testing (see Moser & Stevens, 1992), as concerns testing variance homogeneity, Easterling & Anderson, 1978, and Schucany & Ng, 2006, for testing normal distribution. Nevertheless in applied statistics pre-testing is often applied.

Unfortunately, statistical program packages, lecture notes and applied statistical text books still recommend a pre-test at least on variance homogeneity in the two-sample location problem. If we google for "variance homogeneity test" (24^{th} Sept, 2008) a note is as follows:

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At the latter link, note that the F-test (later in the text alternatively the Levene-test and Mauchly´s test) is recommended as a pre-test in the package XLSTAT. If the F-value is small enough (a table of critical values is given), then it is considered safe to use the t-test.

Lecture notes and text books are recommended for this topic. Nothing is said about what to do if variances are not equal. But this is done under

http://www.sam.sdu.dk/~nks/St2006uk/Variansanalyse-UK.pdf.

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The equivalent of the latter test in our two-sample problem is the Wilcoxon-(Mann-Whitney-) U-test (Wilcoxon, 1945; Mann & Whitney, 1947) which, as a matter of fact, assumes equal moments higher than the first one if the location parameters are to be compared.

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We found in our Google-search more than 500 notes, and most of them recommend pre-tests as concerns the assumption of variance homogeneity.

Pre-testing

Pre-testing means that before the decision between the two hypotheses is made by the test, a researcher tests the assumptions about the distribution using the observations of the random sample(s). Doing so, the overall risk of erroneous decisions is difficult to specify that concerns the tested assumptions and the tested null-hypothesis in question.

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Only if consecutive, independent sampling were applied for both kinds of statistical tests (the pre-test(s) on the one side and the test of H_0 on the other side), could this overall risk of erroneous decisions be calculated using the multiplication rule of probability theory.

However, for reasons of feasibility, just a single sampling of data occurs, meaning that the pre-test(s) and the main test are applied at the same observations. As a consequence, the over-all risk can - due to the dependency of the different test statistics - difficult be calculated in closed form.

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As a pre-test of normality, we use the Kolmogorov-Smirnov-test (Kolmogorov,1930; Smirnov, 1939) and as a pre-test of variance homogeneity the Levene-test (Levene, 1960; this because according to Rasch & Guiard, 2004, the F-test is very sensitive against non-normality and has already been replaced in SPSS).

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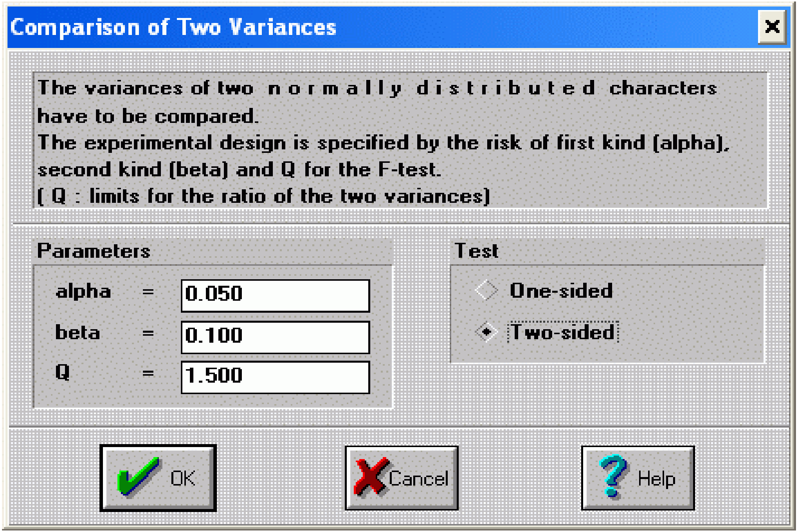
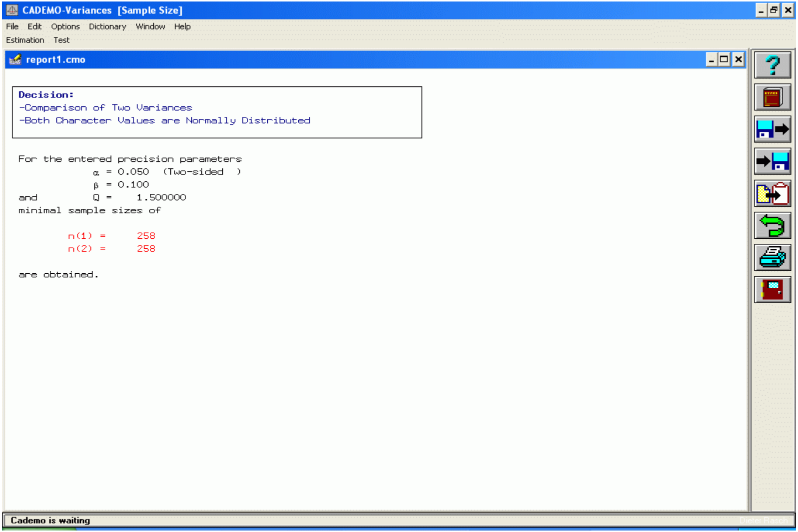
For the Kolmogorov-Smirnov-tests (normality of each distribution) the sample size is difficult to calculate. But we know that it is relatively large $\rightarrow$ 500.

Sample size for comparing 2 variances



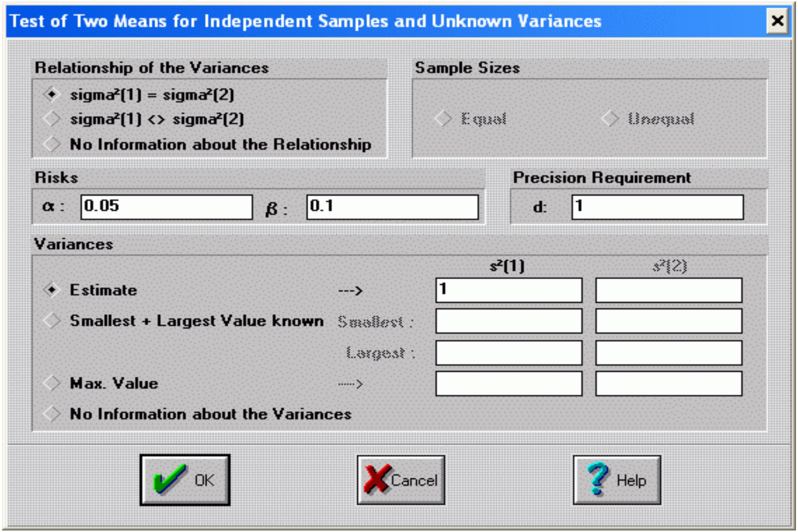
Results (sample size for comparing 2 variances)



Which sample size for pre-tests?

For the F-test (equality of variances) the sample size is calculated by CADEMO for a variance ratio of at least 1.5. We need 258 observations in each sample.

Sample size for comparing 2 means



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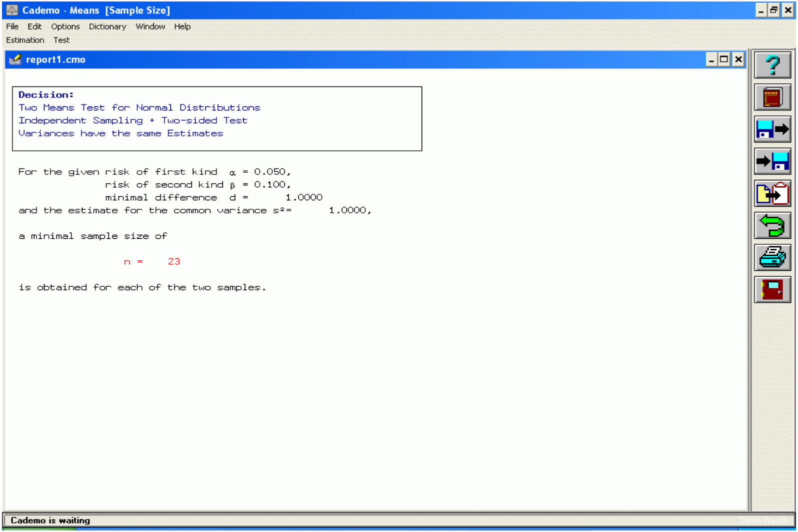

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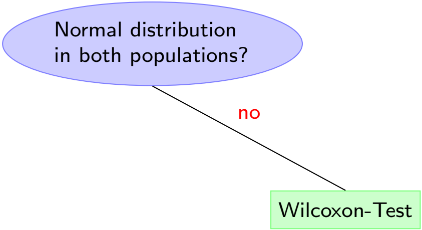
- 258 units for testing homogeneity of variances
- 23 units for testing equality of means

What a nonsense!!

Test-algorithm

Normal distribution
in both populations?

Test-algorithm



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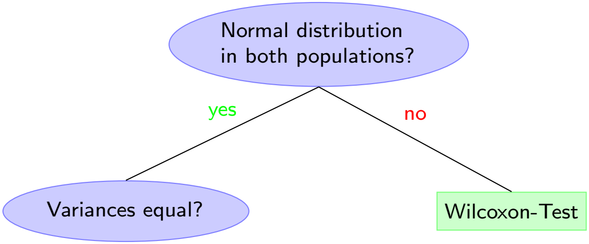

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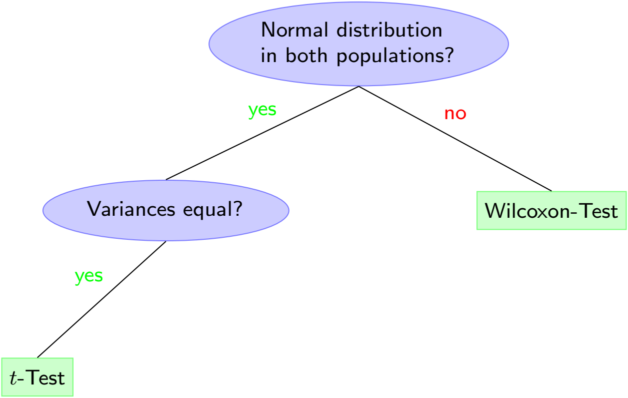

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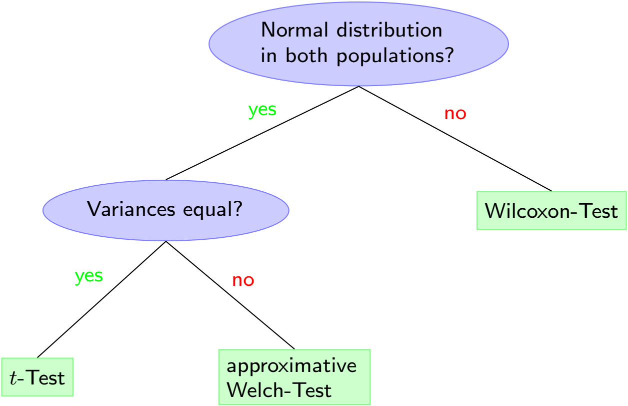

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In the case "always t – Test" is the actual first kind risk α_{act}

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- "near to" the nominal one α_{nom}?
- "near to" means

$$|\alpha_{act} - \alpha_{nom}| \leq 0{,}2\alpha_{nom}$$

Skewness - Kurtosis

For the standardised 3. (Skewness) and 4. moment -3 (Kurtosis) of any distribution we have:

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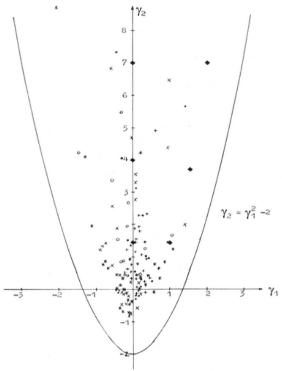

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Simulation experiment

- Each test was done 100 000 times with simulated data. The relative frequency of rejecting H_0 is an estimate of the power function.

Simulation - parameter

We selected:

Type	Skewness	Kurtosis
0	0	0
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2	0,5	15
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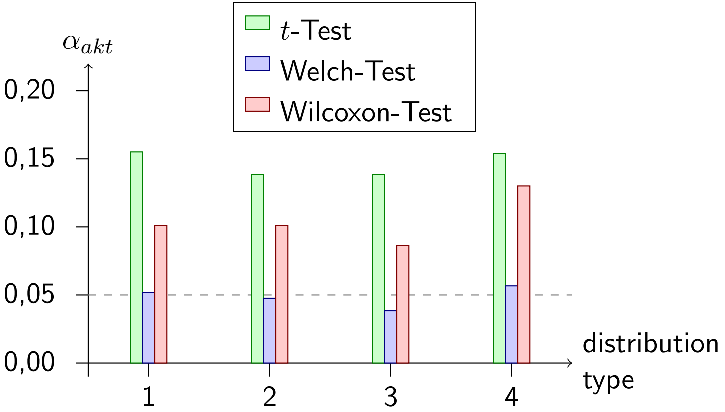
Graphic of specific results (without pre-testing)



Empirical risk of the 1. kind for $t-$, Wilcoxon-, and Welch-test if H_0 is true and α_{nom} $=$ 0.05. The ratio of standard deviations for the first and second sample (σ_1, σ_2) are in ratio 1:2 (n_1=30, n_2=10).

Graphic of specific results (pre-testing)

Empirical risk of the 1. kind for $t-$ and Wilcoxon-test with pre-testing if H_0 is true and $\alpha_{nom} = 0.05$. The ratio of standard deviations for the first and second sample (σ_1, σ_2) are in ratio 1:2 (n_1=30, n_2=10).

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- The t-test can also not be recommanded.

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